Analysis and Optimization
of a
Capacitive Pressure Transducer

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OUTLINE

SHORT SUMMARY

1. Three dimensionless ratios, Zo/Smax, re /re, and Cs /Co , completely specify the electrical parameters of the sensor, i.e., normalized capacitance C/Co , linearity, sensitivity, and curvature of the output voltage response. The definitions of re, rd, and Smax are illustrated in Figure 1. Zo is the deflection of the diaphragm, Cs is the input capacitance of the electronics and Co is the cell parallel plate capacitance realized when the input pressure is zero.

2. Increasing the radii ratio re/rd, of the present design to 0.5, and suitably readjusting other ratios, will concurrently increase the capacitance and reduce the cell size while maintaining linearity and sensitivity constant.

3. Temperature drift of the cell is more than an order of magnitude less than the associated electronics.

INTRODUCTION

The more stringent automotive emissions standards have developed a need for sophisticated control systems, some of which require a manifold absolute pressure transducer. One such transducer, manufactured by Kavlico (http://www.kavlico.com/) uses a pressure sensitive capacitor. The capacitor determines the frequency, f, of a variable oscillator. The output of the variable oscillator and that of a reference oscillator of frequency, fo is translated into an averaged DC voltage of magnitude Vo (1 - f/fo) with the use of a digital gate.* Identical twin oscillators are used ensuring a constant frequency ratio over temperature and minimal voltage drift. This DC output is level shifted and amplified by an op amp.

The purpose of this paper is twofold: (1) To develop the design equations for the capacitive pressure transducer and (2) to use these equations to find the optimum (i.e., lowest cost) design.

I. **GENERAL CONSTRUCTION DETAILS**

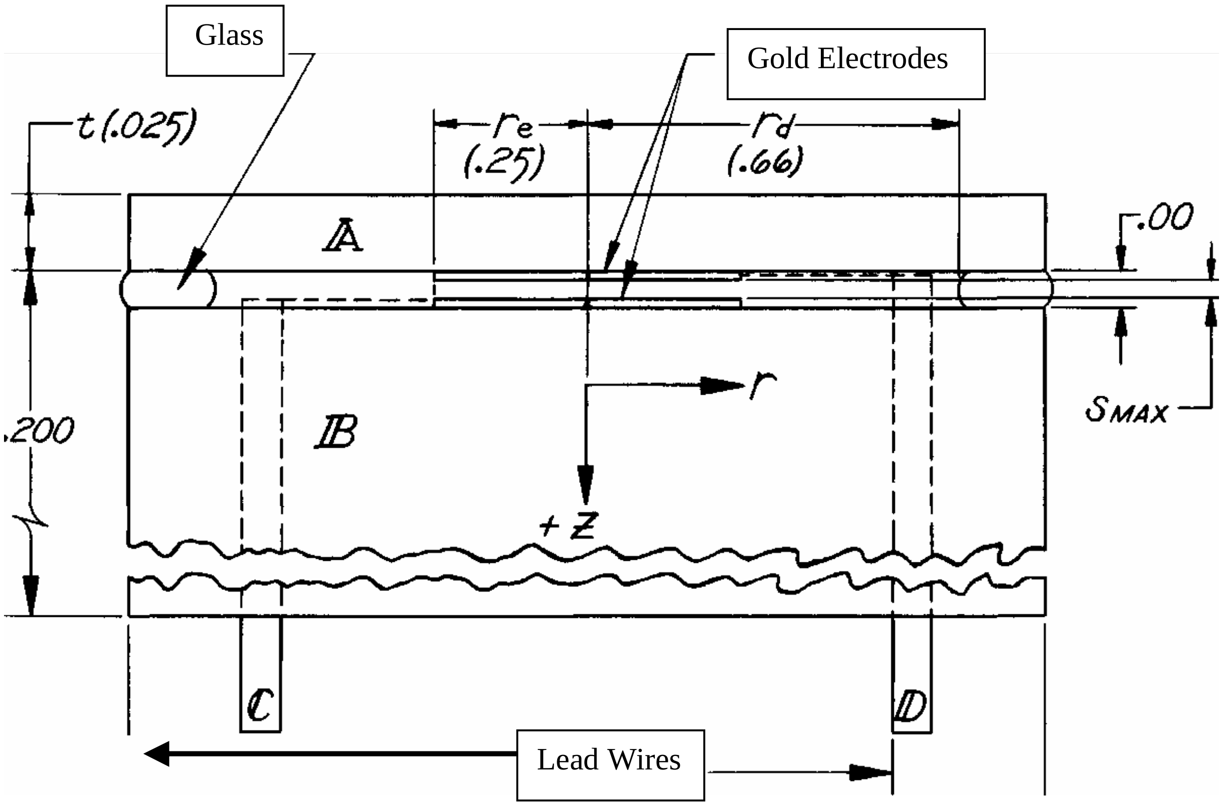


FIGURE 1
CROSS SECTION OF CAPACITOR

The capacitive pressure transducer (PTD) is constructed from two circular plates of polished alumina ("A" and "B") separated by a ring of glass at their periphery. The space between the plates is evacuated and then sealed off through a port on the underside of "B". Gold electrodes on the inner faces of the plates form a capacitor. The gold electrodes and the lead wires are connected by thick-film runners. A pressure difference, δP, across the thin top plate "A" causes it to deflect downwards resulting in an increase of capacitance.

The maximum spacing between the electrodes, Smax occurs when the pressure difference across the top plate goes to zero. Use of the parallel plate capacitor formula and extrapolating capacitance versus pressure data to δP = 0 gives a value of Smax of 1 mil for a Kavlico pressure transducer. Edge effects shall not be considered in this particular analysis as it contributes at most 0.5% to the total value of the capacitance. (Good design suggests use of annular peripheral field electrodes to reduce this component.

II. DEFLECTION UNDER PRESSURE

Downward deflection of the thin upper plate is opposed by a torque resulting from the bond between the peripheral glass and the bottom side of the thin plate. For a rigid material such as glass and because of the relatively small deflection under consideration, it is reasonable to assume that dZ/dr = 0 at the circumference of contact between the glass and ceramic. The primary region of interest for our analysis is the central region of the plates where the greatest contribution to capacitance under bending is made. The nature of the solution of partial differential equations is such that boundary conditions have minimal effects of regions far removed from the boundary. Thus, any minor deviations from our assumed boundary conditions should be of very slight, if any, consequence. The greatest effect of any such deviation would be to lessen the opposing torque at the edges resulting in a greater deflection. Normalizing the calculated deflection to the actual deflection would thus completely remove any such difficulty. The considerable change in internal volume that can occur within a capacitive PTD would result in an accompanying change in internal pressure for a sealed gas filled cavity. This, obviously, is not a consideration for the evacuated PTD.

The solution to the problem of pure bending of a circular plate with clamped edges under uniform applied pressure is given in a number of treatises on elasticity.[1, 2, 3, 4, 5, 6]. The deflection is $Z = zo\,(1 - (r/rd)^2\,)^2$ where zo is the deflection at r = 0

$$z_o := \frac{3\cdot\left(1 - v^2\right)\cdot r_d{}^4\cdot P}{16\cdot Y\cdot t^3}$$

where ν is Poisson's ratio and Y is Young's modulus. Note the following:

1. The deflection is directly proportional to δP.
2. An inflection point occurs at r/rd = 0.577.
3. Toward the center of the plate (where the greatest contributions to capacitance are made) the linear and square terms in r/rd are dominant over cubic and quadric terms.
4. The deflection changes greatly with small changes in the diameter r, or thickness of the top plate. For small changes $\delta z/z = 4\,\delta rd/rd - 3\,\delta t/t$ and, thus, considerable tolerance buildup is virtually unavoidable. Plate thickness is less controllable than diameter so it is a design variable of greater concern. A 20% decrease in thickness nearly doubles the deflection.

The design of the PTD sensor is such that the ratio of maximum deflection to the thickness of the plate is about 0.02. For such small deflections, it is reasonable to assume that, relative to other errors, the deflection is perfectly linear with respect to pressure The excellent linearity of piezoresistive PTD's having similar deflecting diaphragms tends to support this.

Using the parameters for the Kavlico PTD of $Y = 47 \times 10^6$ psi, V = 0.22, t = 25 mil, and r, = 0.66", and assuming δP = 14.7 psi gives a maximum deflection of Zo = 0.67 mil. Various measurements suggest that Zo @ 1 atmosphere for a Kavlico PTD is 0.5 mil or less. More precise and controlled measurements of the various parameters are needed to discern the origin of the discrepancy in Z values.

III. DERIVATION OF CAPACITANCE VERSUS DEFLECTION
Consider a cross section of an annular element of the PTD.

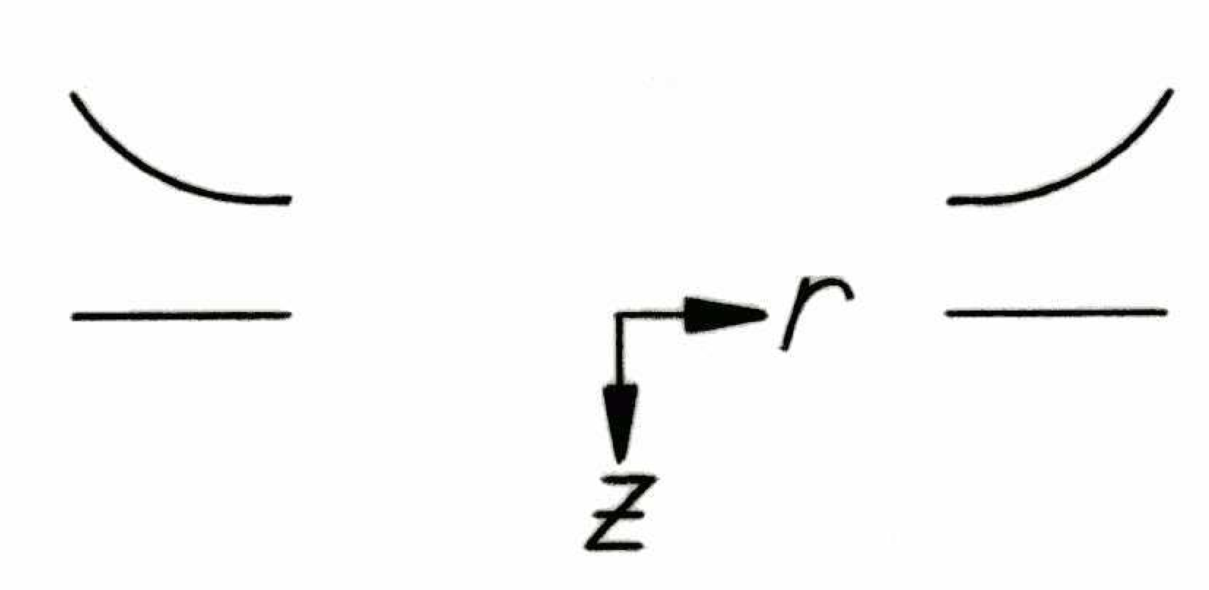


FIGURE 2

Gausses law demands that the local electric field be perpendicular to the surface of a conductor. Thus, the electric field at the upper electrode has both vertical, 1_z and radial 1_r components. The maximum value of the 1_z component occurs at the point of greatest slope which is at the inflection point. The tangent of the angle to the horizontal there is dZ/dr @r = 0.577 rd, which equals 2.28 Zo/rd. Assuming a maximum value of 0.5 mil for Z gives a maximum angle of < 0.01° for a Kavlico PTD. Thus, we can ignore the radial component and still maintain an inaccuracy on the order of 0.1%. We shall assume that the electric field has only a vertical component. Gausses law gives the surface charge density at the electrodes of $\sigma = \varepsilon_o V/S$ where V is the applied voltage and S is the separation between the plates. Then

$$C_{PTD} = \frac{Q}{V} = 2\pi\cdot\varepsilon_o\cdot\int_0^{r_e} \frac{r}{S_{max} - z_o\cdot\left[1-\left(\frac{r}{r_d}\right)^2\right]^2}\,dr$$

Let

$$\xi = \sqrt{\frac{z_o}{s_{max}}}\cdot\left[1-\frac{z_o}{s_{max}}\cdot\left(\frac{r}{r_d}\right)^2\right]$$

then

$$C_{PTD} = \frac{\pi\varepsilon_o\, r_d^2}{\sqrt{Z_o S_{max}}} \int_{\sqrt{Z_o/S_{max}}\,\left(1 - (r_e/r_d)^2\right)}^{\sqrt{Z_o/S_{max}}} \frac{d\xi}{1 - \xi^2}$$

$$C_{PTD} = \frac{\pi\varepsilon_o\, r_d^2}{\sqrt{Z_o S_{max}}} \left[\tanh^{-1}\sqrt{\frac{Z_o}{S_{max}}} - \tanh^{-1}\sqrt{\frac{Z_o}{S_{max}}\left[1 - \left(\frac{r_e}{r_d}\right)^2\right]} \right] \qquad (5)$$

<u>or</u> for $X<1$, $\tanh^{-1} X = 1/2 \ln \frac{1 + X}{1 - X}$ then

$$\frac{C_{PTD}}{C_o} = \frac{1}{2}\left(\frac{r_d}{r_e}\right)^2 \sqrt{\frac{S_{max}}{Z_o}}\; \ln \left[\frac{1 + \dfrac{(r_e/r_d)^2}{\sqrt{\dfrac{S_{max}}{Z_o} - 1}}}{1 - \dfrac{(r_e/r_d)^2}{\sqrt{\dfrac{S_{max}}{Z_o} + 1}}} \right] \qquad (6)$$

where $C_o = \frac{\varepsilon_o \pi r_e^2}{S_{max}}$, $C \to C_o$ as $\Delta P \to 0$

and

$$Z_o = \left(\frac{\text{Center Deflection @ 1 ATM}}{101.325 \text{ KPA}}\right) \Delta P(\text{KPA}) .$$

The derivative of the PTD capacitance of Equation (6) with respect to deflection Z_o is given below.

$$\frac{dC_{PTD}}{dZ_o} = \frac{C_o}{2Z_o}\left(\frac{r_d}{r_e}\right)^2 \left[\frac{1}{1 - Z_o/S_{max}} - \frac{1 - (r_e/r_d)^2}{1 - (Z_o/S_{max})(1 - (r_e/r_d)^2)^2} \right] - \frac{C_{PTD}}{2Z_o} \qquad (7)$$

This result will be needed in succeeding sections. The value of the derivative for the Kavlico design, C_o = 44 pFd, r_e/r_d = 0.38 and Z_o/S_{max} = 0.5, is 120 pFd/mil. The value at Z_o/S_{max} = 0.05 is 41.66 pFd/mil.

Below is a plot of Equation (6) which displays normalized capacitance versus normalized displacement for curves of constant r_e/r_d. The parameters Z_o/S_{max} and r_e/r_d completely specify the normalized capacitance. The nature of the ideal curve is discussed in the next section.

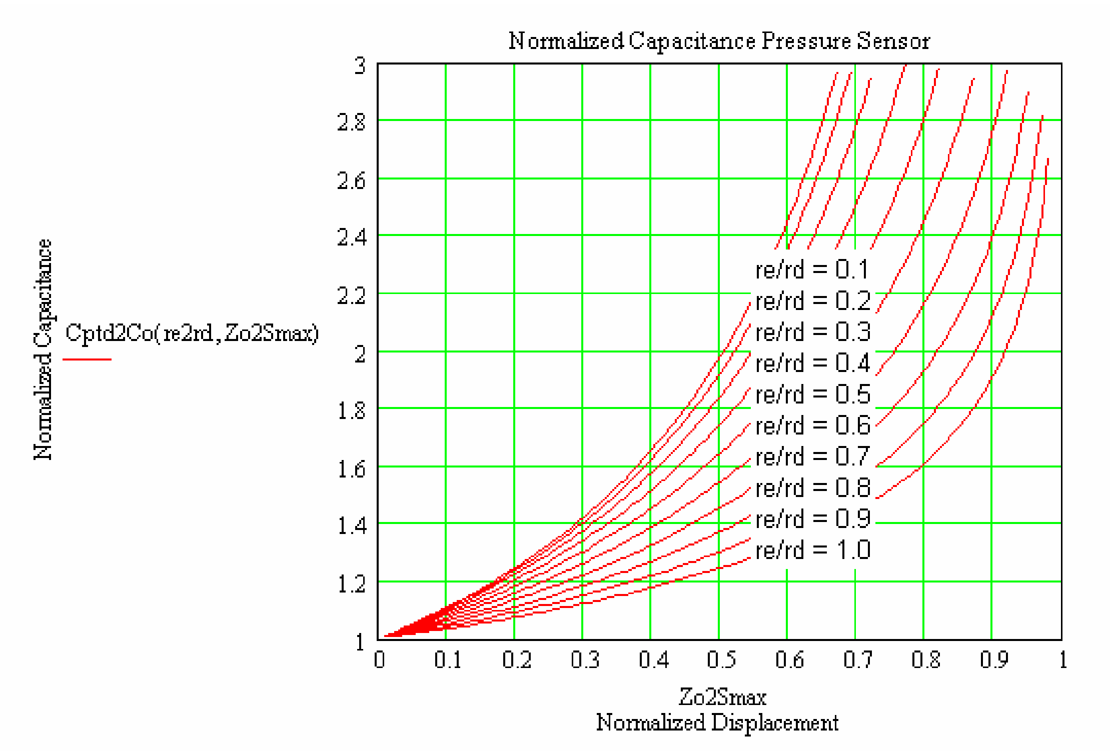
CALCULATE NORMALIZED CAPACITANCE

Rearranging the terms for C_{PTD} in equation (5) and dividing by Co, gives the ratio of C_{PTD} to Co, designated as Cptd2Co. Designate the variables re/rd as re2rd and Zo/Smax as Zo2Smax.

$$\mathrm{tanf}(re2rd, Zo2Smax) := \mathrm{atanh}\left(\sqrt{Zo2Smax}\right) - \mathrm{atanh}\left[\sqrt{Zo2Smax}\cdot\left(1 - re2rd^2\right)\right]$$

$$Cptd2Co(re2rd, Zo2Smax) := re2rd^{-2}\cdot\sqrt{\frac{1}{Zo2Smax}}\cdot \mathrm{tanf}(re2rd, Zo2Smax)$$

$$C_o(re2rd) := \frac{e_o\cdot\pi\cdot\left(r_d\cdot re2rd\right)^2}{S_{max}} \qquad Cs2Co(re2rd) := \frac{C_s}{C_o(re2rd)}$$



V. ERROR ANALYSIS - NONLINEARITY

The nature of the deviation of the capacitance of an actual sensor from the ideal can be discerned by judiciously expanding the capacitance equation (6) into the following infinite series:

$$\frac{C_{PTD}}{C_o} = \frac{1}{1-\frac{Z_o}{S_{max}}} - \frac{\left(\frac{r_e}{r_d}\right)^2 \sqrt{\frac{Z_o}{S_{max}}}}{2\left(1+2\frac{Z_o}{S_{max}}\right)} + \frac{4\left(\frac{r_e}{r_d}\right)^4 \frac{Z_o}{S_{max}}}{3\left(1-\frac{Z_o}{S_{max}}\right)^2} - \frac{\left(\frac{r_e}{r_d}\right)^6 \left(\frac{Z_o}{S_{max}}\right)^{3/2}}{4\left(1+2\frac{Z_o}{S_{max}}\right)^2} + \ldots \qquad (10)$$

The first term on the right hand side in the above expansion is the idealized response. The remaining terms reduce the value of capacitance from the ideal and introduce nonlinearities. Figure 3 and the expansion reveal that r_e/r_d must be made small if the nonlinearities in frequency are to be minimized. This series expansion provides insight into the nature of the deviation from ideal behavior; but because of its complexity, it is not well suited for calculating frequency or voltage nonlinearity.

The deviation of the capacitance of an actual sensor from the ideal can be specified exactly in terms of some infinite series $C_I - C_{PTD} = C_o \sum_{i=1}^{\infty} \alpha_i \left(\frac{Z_o}{S_{max}}\right)^i$. Substituting this into Equation (8), suitably manipulating the resultant equation, and separating linear from nonlinear terms gives

$$\text{Normalized Nonlinear Voltage Component} = \frac{V\left(\frac{Z_o}{S_{max}}\right)}{V_{Reference}} - \frac{a\, Z_o/S_{max}}{1 + C_s/C_o} \qquad (11)$$

where, for convenience, we have defined $a \equiv 1 - \alpha_1$. The last term on the right of Equation (12) is the linear portion of the voltage response of the PTD. The first term on the right hand side can be calculated from Equation (8). Differentiating both sides of the above series equation and evaluating this result at

$Z_o/S_{max} = 0$ gives $1 - \left.\frac{d\left(\frac{C_{PTD}}{C_o}\left(Z_o/S_{max}\right)\right)}{d\left(Z_o/S_{max}\right)}\right|_{Z_o/S_{max}=0} = \alpha_1$. The derivative given in Equation (7) is not very useful in this situation as it is indeterminate at the origin. We define a function, a, of (r_e/r_d) given by

$$a = \lim_{Z_o/S_{max} \to 0} \frac{\frac{C_{PTD}}{C_o}\left(\frac{Z_o}{S_{max}}\right) - 1}{Z_o/S_{max}} \tag{12}$$

The function a is readily calculated from the above to a high degree of accuracy by using a non-zero point infinitesimally close to the origin (Z_o/S_{max} = 0.001 gives satisfactory accuracy). Values of "a" which are useful in calculating linearity are tabulated below:

r_e/r_d:	0.1	0.2	0.3	0.38	0.4	0.5	0.6
a :	0.9897	0.9614	0.9137	0.8641	0.8493	0.7715	0.6837

Curve fitting a function to the tabulated values of a gives the following expression which is accurate to within a few percent: $a = 1 - \left[1 - 0.35\,(r_e/r_d)^2\right](r_e/r_d)^2$.

Nonlinearity of a curve is defined as the percent error band of nonlinear terms about the linear portion of the curve within some specified operating range. Nonlinearity can be calculated as one-half of the nonlinear component divided by the linear component of a curve. Using the results of Equations (11) and (8) gives

$$\begin{matrix}\pm\ \%\ \text{nonlinearity} \\ \text{of output voltage}\end{matrix} = 100\left(\frac{(C_{PTD}/C_o - 1)\ (1 + C_s/C_o)}{2a\ (C_{PTD}/C_o + C_s/C_o)\ Z_o/S_{max}} - \frac{1}{2}\right) \tag{13}$$

for nonlinearity over the operating range $\Delta P = 0$ to $\Delta P\ (Z_o/S_{max})$.

Frequency is a decreasing function of capacitance, and thus output voltage is an increasing function of capacitance. Therefore, adding a stray capacitance, C_S, to the initial capacitance, C_o, causes a positive displacement of voltage, $C_s/(C_o+C_s)$. This offset in voltage at the origin can be nulled out by equally offsetting the origin and recognizing that the initial capacitance is $C_o + C_s$. This is the approach adopted for Equations (8) and (13).

The stray capacitance offset, of course, has not disappeared but has been shifted to outer lying values of the coordinate Z_o/S_{max}. Has nonlinearity disappeared at the origin? This question is answered by taking first and second derivatives of Equation (8), obtaining the expression for the curvature of the output transfer function,

$$\frac{1}{V}\ \frac{d^2V}{d(Z_o/S_{max})^2} = \frac{C_o + C_s}{(C_{PTD} + C_s)^2}\ \frac{d^2C_{PTD}}{d(Z_o/S_{max})^2} - \frac{2(C_o + C_s)}{(C_{PTD} + C_s)^3}\left(\frac{d\ C_{PTD}}{d\ Z_o/S_{max}}\right)^2$$

For the ideal V = Zo/Smax straight line response in the absence of stray capacitance, Cs , the curvature vanishes, i.e., the above expression equals zero. Introducing stray capacitance causes the greatest reduction in the last term of the above expression giving a positive curvature, i.e., the voltage response bows upward. The greatest relative influence of Cs occurs when C_{PTD} is smallest, i.e., at the origin where $\delta P = 0$. For an actual non-ideal sensor, Figure 3 reveals that relative to the ideal, the C_{PTP}/Co slope is decreased and its curvature increased, causing the above equation and the voltage curvature to increase positively or bow upward. Thus, unfortunately, non-deal and stray capacitance induced curvatures are additive. However, because the actual PTD capacitance is less than the ideal for some nonzero displacement, the non-ideal voltage response is shifted downward relative to the ideal, bucking the sign of the stray capacitance-voltage offset.

The original question concerning nonlinearity at the origin is semantically incorrect. Linearity is not defined at a point but over a specified range. Secondly, nonlinearity is not influenced by offset, it is only a function of interval size and curvature. A finite offset can always be nulled out while curvature is unaffected by a linear amplifier. Thus, the bucking of stray and non-ideal offsets does not improve linearity. Non-linearity is not the same as curvature. For large stray capacitance, curvature at the origin is large. By taking a sufficiently small interval, nonlinearity can be made arbitrarily small.

VI. DESIGN BY OPTIMIZATION

When regarding the PTD system as a black box, the only observable sensor parameter is nonlinearity. In what follows, it will be seen that juggling the design variables to comply with some specification on non-linearity over a pressure range while simultaneously trying to make the sensor as small as possible is the major task of optimizing the design.

All of the electrical parameters of the PTD sensor C_{PTD}/Co, V/V_{Ref}, d^2V/V, and nonlinearity are functions of dimensionless ratios viz. re/rd, Zo/Smax and Cs /Co. Mathematically, the factors of the ratios are free to assume any values; the real world, however, imposes some restraints. Because of the influence of Cs /Co on nonlinearity, minimum stray capacitance Cs is a crucial parameter. The nominal spacing between electrodes Smax is also of importance. Technological limitations are such that Smax ~ 1 mil is probably the minimum practical value for the max spacing. The electronics also imposes some constraints. From a signal processing point of view, it is desirable to have sensitivity as large as possible. Small values of C_{PTD}/Co @maximum δP would certainly compromise the signal to noise ratio of the system. Thus, some specified C_{PTD}/Co at maximum pressure is a third constraint.

Summarizing the design situation: There are eight variables, Zo, Smax, re, rd, t, Cs, Co and C_{PTD}. There are four equations: C_{PTD}, Co, Zo, and NonLinearity. There are three constraints: Cs, Smax, and Sensitivity. Eight variables and seven conditions leave one degree of freedom. First we will implement the existing constraints. Then we shall use the last degree of freedom to optimize the system by specifying a price/performance figure of merit, EFM.

Economics dictates that the best design is the lowest cost design that meets all specifications. Thus, **minimizing the radius of the plate**, rd, which will give the **lowest material cost**, is the condition for optimum design. This requirement removes the last degree of freedom and uniquely specifies the design. The plate radius is given by

$$r_{plate} = \sqrt{\frac{S_{max} \cdot C_s}{\pi \cdot \varepsilon_o}} \cdot \left(\sqrt{\frac{C_s}{C_o}} \cdot \frac{r_e}{r_d} \right)^{-1}$$

Where Smax and Cs are given constants (which effectively scale the size of the transducer) and Cs /Co and re/rd are dimensionless ratios. The goal of optimization is to find the design variables Zo/Smax, re/rd, and Cs/Co that meet the given specifications and yield the maximum value of

$$\sqrt{\frac{C_s}{C_o}} \cdot \frac{r_e}{r_d}$$

which is an economic figure of merit (EFM) for the system. The optimum design parameters yield the greatest EFM.

For example, given the design requirements, C_{PTD}/Co = 1.35 at maximum pressure, δP and + - nonlinearity = 1.5%, the following trial solutions were obtained by the above procedure:

MATHCAD OPTIMIZATION SOLUTION:

Solve Equation (5) For Zo/Smax Given Re/Rd And The Required Sensitivity

Estimate zo = 0.5

$$z(re2rd) := root(Cptd2Co(re2rd, z0) - Sen, z0)$$

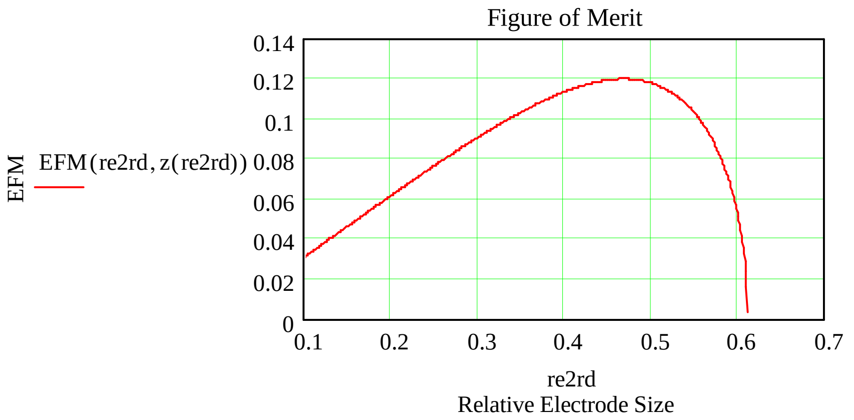
Set Design Constraints On Sensitivity And Linearity:

$$Cs2Co(re2rd, Zo2Smax) := \frac{Cptd2Co(re2rd, Zo2Smax) \cdot (1 - 2 \cdot a(re2rd) \cdot A \cdot Zo2Smax) - 1}{1 + 2 \cdot a(re2rd) \cdot A \cdot Zo2Smax - Cptd2Co(re2rd, Zo2Smax)}$$

Maximize sensitivity and mimimize cost by maximizing figure of merit, EFM:

$$EFM(re2rd, Zo2Smax) := re2rd \cdot \sqrt{Cs2Co(re2rd, Zo2Smax)}$$

Plot Figure of Merit, EFM:



$$\mathrm{Dir}(re2rd) := \frac{d}{d\,re2rd} EFM(re2rd, z(re2rd))$$

ROpt := 0.5 Given $|\mathrm{Dir}(ROpt)| = 0$

Minerr(ROpt) = 0.468

For specified conditions, select re/rd = **0.468 for optimum**.

REFERENCES

1. G. Nadeau, "Introduction to Elasticity", 1964, p. 186.
2. E. Mansfield, "The Bending and Stretching of Plates", 1964, p. 38.
3. "Treatise on the Mathematical Theory of Elasticity", 1944, p. 475.
4. Landau - Lifshitz, "Theory of Elasticity", 1959, p. 50.
5. Temashenko, "Theory of Plates and Shells", 1959, p. 55.
6. Samaun, "An Integrated Circuit Piezoresistive Pressure Sensor for Biomedical Instrumentation", 1971 Standford Univ., Ph.D. Thesis, p.45.